Sketching Quadric Surfaces Made Easy

By Pat Rossi

Introduction

During the past two decades, the proliferation and availability of mathematical software has created a situation in which both math faculty and students of multi-variable calculus have the opportunity to use graphing software to graph quadric surfaces and to use these graphs in the solution of problems involving double and triple integrals.

For students who have already mastered the drawing of quadric surfaces, such software can be a great time saver, allowing them to focus entirely on concepts and problem-solving, rather than spending their time and energy in the time-consuming activity of drawing three-dimensional objects on a two-dimensional surface. However, for those students who have not yet mastered the art of drawing quadric surfaces, it is quite likely that when they view a computer generated image of such a surface, they will not recognize the mathematical significance of what they are seeing. Simply put, they won't be able to associate the distinguishing features of the image with the characteristics of the equation from which the image is derived. There we have it — the introduction of students to the use of new software (coupled with the unrealistic expectation that they will be able to correctly interpret the images generated by the software), at exactly the moment when they are gaining their first exposure to the difficult topics of quadric surfaces and multiple integrals. Terrible timing, to say the least!

Thus, it becomes obvious, that students of multi-variable calculus still need to learn how to draw quadric surfaces. In this paper, we consider some techniques and observations that can be passed on by instructors to their students with little, if any, extra investment of time. Most, if not all, of these techniques and suggestions can be seamlessly integrated into the examples that the instructor presents while teaching the topics of quadric surfaces and multiple integrals.

Before beginning our discussion, we acknowledge two things. First, in this paper, we adhere to the convention of drawing the y and z axes "on the page," with the y-axis being horizontal, the z-axis being vertical, and the x-axis "coming out of the page."

Second, since almost all calculus texts make mention of such things as the techniques of sketching "level curves" and graphing the "traces" of a graph in the coordinate planes, we will assume that the reader is familiar with these, and will not mention them here.

Relationships That Are Preserved - And Those That Are Not

First of all, we acknowledge that not all geometric relationships which exist in a three-dimensional setting are preserved when a three dimensional object is drawn on a two-dimensional surface. Although some instructors of calculus are apparently oblivious to this fact, it becomes immediately obvious when we observe that when the coordinate axes are drawn, the x-axis cannot be drawn perpendicular to either of the other two coordinate axes. That having been said, it's worth our while to consider which geometric relationships are preserved when drawing a three dimensional object on a two dimensional surface, and which relationships are not preserved.

Perpendicularity - In general, lines and surfaces that are perpendicular in three dimensions cannot be drawn perpendicular on a two dimensional surface. The exception is that perpendicularity in the y-z plane is preserved. (i.e., lines (and curves) in the y-z plane that are perpendicular in 3-D can be drawn perpendicular on a two dimensional surface.) For an obvious example, consider the y and z axes.

Intersection - Lines and surfaces that intersect in three dimensions also intersect when drawn on a two dimensional surface.

Parallelism - Parallel relationships are preserved. Lines, curves, and surfaces that are parallel in three dimensions are parallel when drawn in two dimensions.

Tangency - Tangency is preserved. This becomes obvious when one realizes that tangency involves a point of intersection and is, essentially, a parallel relationship. A curve which is tangent to a line L at point p is very nearly parallel to line L in the immediate vicinity of point p. Instructors and students alike often find it difficult to sketch circles and ellipses because they aren't mindful of this fact. (i.e., they do not recognize the necessity of drawing a curve very close to its tangent in the immediate vicinity of the point of tangency.)

To illustrate, a circle of radius r, centered at the origin, is tangent to the lines, $y = \pm r$, and $x = \pm r$ at the points, $(0, \pm r)$ and $(\pm r, 0)$ respectively, and for this reason, is often sketched using a square with sides of length $2r$ centered at the origin. Sadly, even when such a square IS used as a guide for drawing a circle, little, if any, attention is given to drawing the circle tangent to the lines $y = \pm r$, and $x = \pm r$ at, and in the vicinity of, the points, $0, \pm r)$ and $(\pm r, 0)$. The fact is, that excellent results can be obtained by sketching these parts of the circle first, and finishing the sketch afterward. (See Fig. 1)

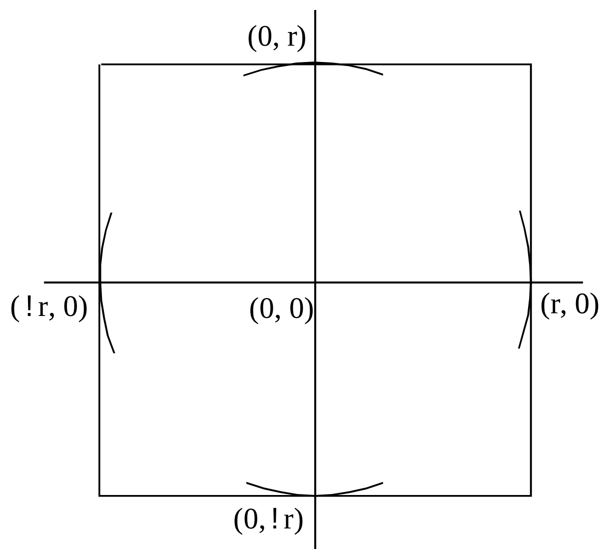

Fig. 1

A similar approach should be used when sketching ellipses, using the fact that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is tangent to the lines $y = \pm b$ and $x = \pm a$ at the points $(0, \pm b)$ and $(\pm a, 0)$ respectively.

Once this sketching technique has been mastered, it can be applied to the sketching of circles and ellipses in a plane other than the y-z plane, in which perpendicular relationships are not preserved. The tangency relationship is our main tool in drawing circles and ellipses, and is probably the only tool that we need to use. (See Fig. 2)

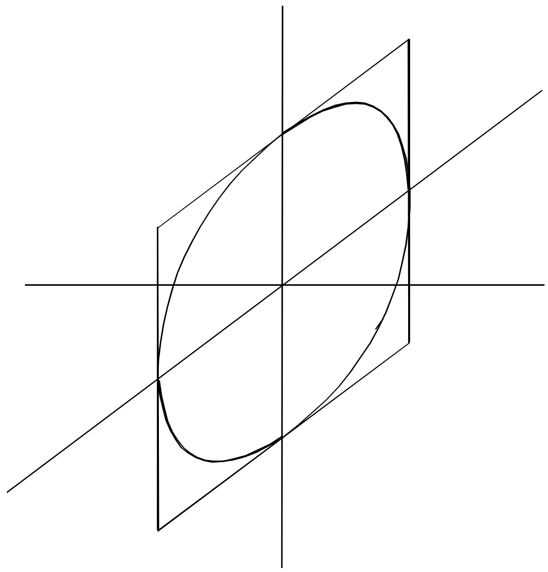

Fig. 2

Our observation regarding the preservation of parallel relationships and the fact that perpendicular relationships are only preserved in the y-z plane can be applied immediately to the drawing of increment marks on the coordinate axes. An elementary mistake made by many students (and some instructors) is that of drawing the increment marks perpendicular to the axis on which they appear, in an attempt to preserve the perpendicular relationship between the axes and their increment marks. This mistake alone does much to destroy the illusion of three dimensions. Increment marks should be drawn in such a way that they are parallel to one of the other two axes, and not necessarily perpendicular to the axis on which they are drawn. In particular, increment marks on the x and y axes should be drawn in such a way that they are contained in the x-y plane. Furthermore, increment marks on the x-axis should be drawn parallel to the y-axis and increment marks on the y-axis should be drawn parallel to the x-axis. Increment marks on the z-axis may be drawn in one of two ways. They can be drawn so as to be contained in the y-z plane and hence, drawn parallel to the y-axis. The increments on the z-axis can also be drawn so as to be contained in the x-z plane and hence, drawn parallel to the x-axis. The way that you choose to draw a surface will dictate how the increments are drawn. (e.g. a trace in the x-z plane will call for increment marks on the z-axis to be drawn parallel to the x-axis, while a trace in the y-z plane will call for increment marks on the z-axis to be drawn parallel to the y-axis. (See Figures 3 and 4)

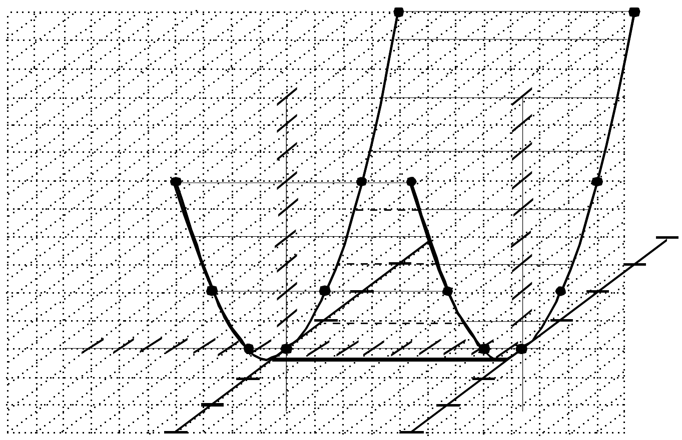

Fig. 3

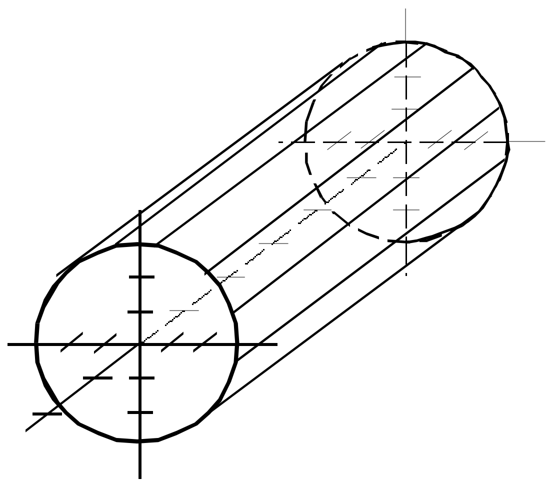

Fig. 4

Another tip regarding increment marks concerns spacing. Spacing between increment marks on the y and z axes should be uniform. There are two options for the spacing of increment marks on the x-axis. One may chose uniform spacing, as is used on the y and z-axes. As an alternative, we can choose a slightly larger scale on the positive x-axis and a slightly smaller scale on the negative x-axis. This technique makes the positive x-axis appear closer than the other axes, and the negative x-axis appear to be farther away, giving the x-axis the appearance of "coming out of the page," and thereby enhancing the illusion of three dimensions. For the sake of comparison, Fig. 5 employs a uniform scale on the x-axis, while Fig. 6 features the use of a non-linear scale with the larger increments appearing on the positive x-axis.

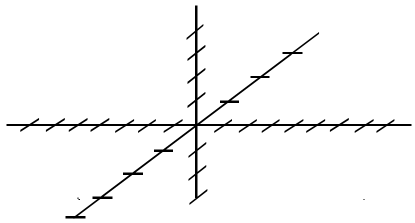

Fig. 5

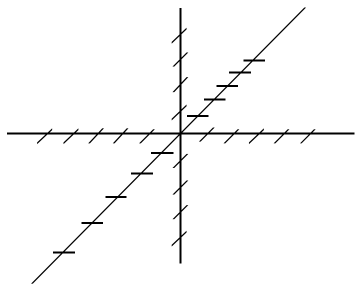

Fig. 6

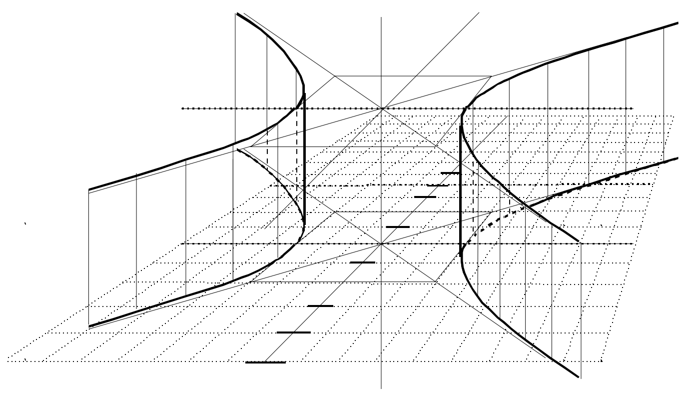

Fig. 7

Figure 7 illustrates how such a non-linear scale can be used effectively to sketch a hyperbolic cylinder.

Symmetry - A consideration of Fig. 2 quickly reveals that a circle or an ellipse drawn in a plane other than the y-z plane is not symmetric. The same is true of parabolas and hyperbolas. (i.e. parabolas and hyperbolas should not be drawn symmetrically about their axes of symmetry in any of the coordinate planes except the y-z plane.) Nevertheless, symmetry should be a consideration when sketching these objects. For example, consider construction of the trace of the graph of $z = x^2$ in the x-z plane. (See Fig. 8) Although the graph is not drawn symmetrically on the page, the *illusion* of symmetry is preserved by plotting points for "strategically chosen," positive values of x, and then plotting their symmetric counterparts on the other side of the axis of symmetry, equidistant from the original points. An important observation to be made concerning this, is that in order to preserve the illusion of symmetry, each pair of points is plotted on a line segment that is parallel to the x-axis. (Equivalently, the vertical distance from each point to the x-axis is measured along a line segment that is parallel to the z-axis.)

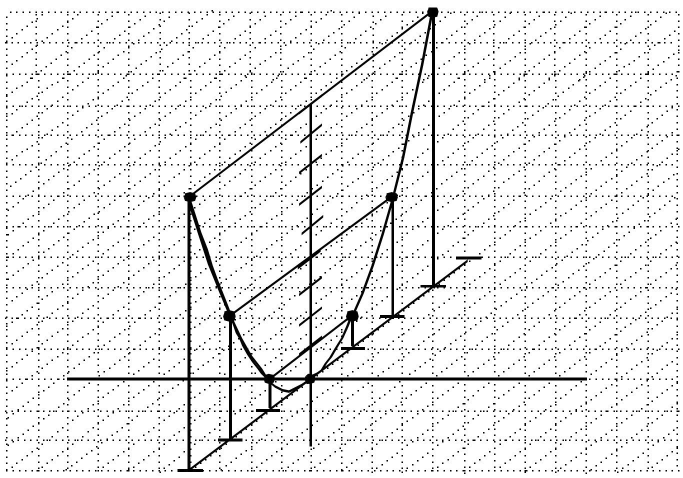

Fig. 8

One final observation that should be made regarding Fig. 8 is the use of a "3-dimensional grid" as an aid in drawing the parabola. All lines in the grid are parallel to one of the coordinate axes, and serve as an aid in helping students draw with the mind-set of preserving parallel and tangential relationships, as well as drawing increment marks parallel to one of the other coordinate axes.

At the instructor's discretion, students can be encouraged to use the grid sheets to do homework problems and test problems that involve the graphing of quadric surfaces. Such grid sheets are easy to make, and can be made available to students either as a download from the internet, or in "notepad form" at the bookstore.

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