An Elementary Proof of the Divergence of the Infinite Inverse Prime Number Series

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Introduction

In this paper, we give an elementary proof of the divergence of the infinite prime number series. The infinite inverse prime series is not mentioned in the usual undergraduate textbooks of either Advanced Calculus or Abstract Algebra. The reason is that the classical proofs are considered to be too difficult and often assume knowledge of some number theory results which are not encountered until graduate school. Here we present a proof that is easy enough for the undergraduate mathematics major to understand. The proof requires knowledge of some Calculus and Abstract Algebra.

Let $M = \left\{ \prod_{i=1}^{r} p_{n_i} : \ p_{n_i} \text{ are prime, with } p_{n_i} \neq p_{n_j} \text{ if } n_i \neq n_j \right\}$, and let $P = \{p \in \mathbf{N} : p \text{ is prime}\}$. The following theorem from Number Theory is well known, and can be found in [1]. We give a proof here for the sake of completeness.

Lemma 1. *If $n \in \mathbf{N}$, then $n = mj^2$ where $j \in \mathbf{N}$ and $m \in M$, and this representation is unique.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha 2} \dots p_i^{\alpha_i}$, where the p_i's are prime, be the prime decomposition of n. Note that α_i can be written as $\alpha_i = 2b_i + r_i$, where $r_i = 0$ or 1, depending on whether α_i is even or odd. Clearly:

$$\begin{aligned} n &= p_1^{2b_1+r_1} p_2^{2b_2+r_2} \dots p_i^{2b_i+r_i} \\ &= p_1^{r_1} p_2^{r_2} \dots p_i^{r_i} \left(p_1^{b_1} p_2^{b_2} \dots p_i^{b_i} \right)^2 \\ &= mj^2 \end{aligned}$$

as desired. □

LEMMA 2. $\sum_{m \in M} \frac{1}{m}$ *diverges.*

PROOF. Fix $k \in \mathbf{N}$. Then:

$$\begin{aligned} \sum_{n \in \mathbf{N},\ n<k} \frac{1}{n} &= \sum_{m \in M,\ j \in \mathbf{N},\ mj^2<k,} \frac{1}{mj^2} \\ &\le \left(\sum_{m \in M,\ m \le k} \frac{1}{m} \right) \left(\sum_{j \in \mathbf{N},\ j \le k} \frac{1}{j^2} \right). \end{aligned}$$

Now, let $k \to \infty$. Since $\sum_{n \in \mathbf{N}} \frac{1}{n}$ diverges, and $\sum_{j \in \mathbf{N}} \frac{1}{j^2}$ converges, it follows that $\sum_{m \in M} \frac{1}{m}$ diverges. □

The next lemma is a simple exercise in Calculus.

LEMMA 3. $e^x > 1 + x$ *for* $x \ge 1$.

PROOF. Let $f(x) = e^x - 1 - x$. Note that $f(1) > 0$ and that $f'(x) = e^x - 1 > 0$ for $x \ge 1$. Therefore, $f(x)$ is increasing for $x \ge 1$, and the result follows. □

THEOREM 1. $\sum_{p \in P} \frac{1}{p}$ *diverges.*

PROOF. Suppose that $\sum_{p \in P} \frac{1}{p}$ converges and let S be its sum. Fix $k \in \mathbf{N}$.

$$\begin{aligned} e^S &> e^{\left(\sum_{p \in P,\ p<k} \frac{1}{p} \right)} \\ &= \prod_{p \in P,\ p<k} e^{\frac{1}{p}} \\ &\ge \prod_{p \in P,\ p<k} \left(1 + \frac{1}{p} \right) \\ &= 1 + \sum_{m \in M,\ m<k} \frac{1}{m} \\ &> \sum_{m \in M,\ m<k} \frac{1}{m}. \end{aligned}$$

Now let $k \to \infty$. Since the last series diverges, we get a contradiction. This finishes the proof. □

References

[1] Ireland, K. and Rosen, M., 1990, A Classical Introduction to Modern Number Theory, Springer-Verlag, New York.

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