Alabama Journal of Mathematics
38 (2014)

ISSN 2373-0404

Forbidding Rainbow Cycles with Edge-Colorings of $K_{m,n}$

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The edges of the complete bipartite graph $K_{m,n}$ can be colored with k colors appearing so that no cycle subgraph is rainbow if and only if $k \in \{1, \ldots, m+n-1\}$. The rainbow-cycle-forbidding edge-colorings of $K_{2,n}$ with $n+1$ colors appearing are completely characterized and counted.

Introduction

For standard notation, terminology, and elementary facts in graph theory, see West (2001) or any of the several excellent graph theory textbooks now available. The only graphs that will play a role here are K_n, the complete graph on n vertices, $K_{m,n}$, the complete bipartite graph with m vertices in one part and n in the other, and C_t, the cycle on $t(\geq 3)$ vertices. An *edge-coloring* of a graph is just what it sounds like, an assignment of "colors" or "symbols" to the edges of the graph, one color to each edge. If G is edge-colored, a subgraph H of G is *rainbow* with respect to that coloring if and only if no two different edges of H bear the same color. If there are no rainbow copies of H in edge-colored G,we say that the edge-coloring of *G forbids* rainbow H.

Most *anti-Ramsey* theory has to do with conditions under which there is an edge-coloring of a graph G which forbids rainbow H, for some H; almost always, G is a complete graph and H is either a smaller complete graph or one of the "usual suspects", such as a cycle or a path. For instance, Gouge, Hoffman, Johnson, Nunley, and Paben (2010) depart from the following, which was well-known long before.

Theorem 1. *Suppose that $n \geq 3$ and $t \geq 1$* are integers. The following are equivalent.

(a) There is an edge-coloring of K_n with t colors appearing which forbids rainbow cycles.

(b) There is an edge-coloring of K_n with t colors appearing which forbids rainbow $K_3(\simeq C_3)$.

(c) $t \leq n-1$.

One of the results in Gouge et al. (2010) characterizes the rainbow-cycle-forbidding edge-colorings of K_n, with $n-1$ colors appearing, sufficiently well that the essentially different such colorings can be counted. The discoveries to be presented here were inspired by the desire to find results parallel to these in Gouge et al. (2010), with $K_{m,n}$ replacing K_n and with C_4, the shortest cycle in $K_{m,n}$, replacing C_3, the shortest cycle in K_n. Perhaps predictably, just because $K_{m,n}$ is a less "dense" graph than K_{m+n}, and therefore the constraint of coloring to avoid rainbow cycles is less constraining in $K_{m,n}$, permitting too rich a variety of colorings for a succinct description, we have not succeeded! But we do have some results of interest; and perhaps someone will see something in these results that we have missed.

Results

Theorem 2. *Suppose that m, n and t are positive integers. The following are equivalent.*

(a) There is an edge-coloring of $K_{m,n}$ with t colors appearing which forbids rainbow cycles.

(b) There is an edge-coloring of $K_{m,n}$ with t colors appearing which forbids rainbow C_4.

(c) $t \leq m+n-1$.

Proof. Clearly (a) implies (b).

Suppose that an edge-coloring of $K_{m,n}$ admits a rainbow cycle. Since the only cycle subgraphs of $K_{m,n}$ are C_{2q}, $q \geq 2$, the rainbow cycle is one of these. If $q > 2$ then there is a *chord* of the rainbow C_{2q}, i.e., an edge joining two vertices on the cycle which are not adjacent on the cycle, which is an edge of $K_{m,n}$. See Figure 1.

The vertices of the chord are the endvertices of two paths on the rainbow C_{2q}. With each path the chord makes a cycle subgraph of $K_{m,n}$ with fewer than $2q$ vertices. At least one of these cycles is rainbow, because the color appearing on the chord cannot appear on both paths, since the C_{2q} is rainbow.

Therefore, the existence of a rainbow cycle of order > 4 in an edge-colored $K_{m,n}$ implies the existence of a shorter

The work in this paper was supported by NSF grant no. 1262930
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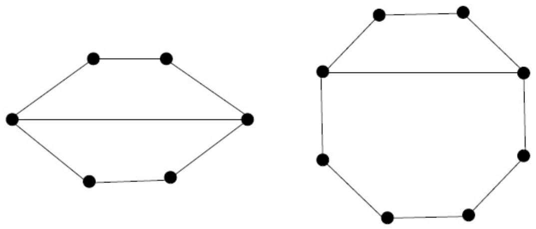

Figure 1. Chords of C_6 and C_8 which are edges of any $K_{m,n}$ of which these cycles are subgraphs.

rainbow cycle. Therefore, the existence of a rainbow cycle implies the existence of a rainbow C_4. Thus (b) implies (a).

Suppose that $t \geq m + n$ and $K_{m,n}$ is edge-colored with t colors actually appearing. Take $m + n$ edges of different colors, and let G be the subgraph of $K_{m,n}$ "induced" by these edges. Then G is a graph with $m + n$ edges on no more than $m + n$ vertices. By a fundamental result of graph theory, G has a cycle subgraph. Since G is a rainbow subgraph of $K_{m,n}$, it follows that $K_{m,n}$ has a rainbow cycle subgraph. Thus (a) implies (c).

To prove that (c) implies (a), clearly it suffices to show that $K_{m,n}$ can be edge-colored with $m + n - 1$ colors so that there is no rainbow cycle subgraph. We proceed by induction on $m + n$. At the beginning, $m = n = 1$ and the claim to be proven is obviously true.

Suppose that $m + n > 1$. Without loss of generality, we suppose that $n > 1$. Applying the induction hypothesis let $K_{m,n-1}$ be edge-colored with $m+n-2$ colors appearing so that there is no rainbow cycle subgraph. Introduce a new vertex to form $K_{m,n}$, and color all m edges incident to the new vertex with a single color, different from the $m+n-2$ colors already appearing. Clearly the result is an edge-coloring of $K_{m,n}$ with $m + n - 1$ colors with no rainbow cycle subgraph. □

If $K_{m,n}$ is edge-colored, we will say that a color c appearing in the coloring is *dedicated* to a vertex v of $K_{m,n}$ if the edges c appears on are all incident to v.

Corollary 1. *Suppose that $K_{m,n}$ is edge-colored with $m+n-1$ colors appearing so that there are no rainbow cycle subgraphs. Then for each vertex of $K_{m,n}$ there is at least one color dedicated to that vertex.*

Proof. Suppose that $K_{m,n}$ is so colored, and that v is a vertex of $K_{m,n}$. If no color is dedicated to v then $m + n - 1$ colors appear on the edges of $K_{m,n} - v \in \{K_{m-1,n}, K_{m,n-1}\}$, which would imply, by Theorem 2, that $K_{m,n} - v$, and thus $K_{m,n}$, contains a rainbow cycle subgraph. □

Corollary 2. *Suppose $n \geq 2$. An edge coloring of $K_{2,n}$ with $n + 1$ colors appearing forbids rainbow C_4's if and only if there is a one-to-one correspondence between the n vertices in one part and a set of n of the colors such that each of the vertices corresponds to a color dedicated to it.*

Proof. If the coloring forbids rainbow C_4's then by Corollary 1 each of the n vertices has at least one color dedicated to it, and clearly no color can be dedicated to two different vertices in the same part, so the one-to-one correspondence exists.

On the other hand, suppose the vertices in one part of $K_{2,n}$ are $v_1, \ldots, v_n$, and the colors appearing in the edge-coloring are $c_0, c_1, \ldots, c_n$, with c_i appearing only on edges incident to v_i, $i = 1, \ldots, n$. Every C_4 in $K_{2,n}$ will contain two vertices v_i and v_j for some $1 \leq i < j \leq n$. If both edges incident to v_i are colored c_i then no C_4 containing v_i is rainbow. Therefore the only way C_4 containing v_i and v_j can be rainbow is if c_i appears on only one of the two edges incident to v_i, and c_j on only one edge incident to v_j. But then c_0 appears on two edges of the C_4, so the C_4 is not rainbow. □

Two edge-colorings of a $K_{m,n}$ with labeled vertices are essentially the same, or equivalent, if a relabeling of the vertices and a renaming of the colors transforms one coloring into the other. If two edge-colorings are not essentially the same, then they are *different*.

Theorem 3. *Suppose that $n \geq 3$. The number of different edge-colorings of $K_{2,n}$ with $n + 1$ colors appearing which forbid rainbow C_4's is $\frac{n(n+4)}{4}$, if n is even, and $\frac{n^2+4n-1}{4}$, if n is odd.*

Proof. Let the parts of $K_{2,n}$ be

$$W = \{w_1, w_2\} \text{ and } N = \{v_1, \ldots, v_n\}.$$

Let the colors appearing in our edge-colorings be $c_0, \ldots, c_n$, with c_i dedicated to v_i, $i = 1, \ldots, n$. Then each v_i has either both edges incident to it colored c_i, or one colored c_i and the other colored c_0. We may assume, after possibly renaming the v_i, that the color c_i appears twice at v_i, $1 \leq i \leq n - p$, and only once at v_i, $n - p < i \leq n$ for some $p \in \{1, \ldots, n\}$; $p \geq 1$ because c_0 has to appear somewhere.

Clearly edge-colorings associated with different values of p are essentially different. We claim that to each value of p there correspond exactly $\lfloor \frac{p}{2} \rfloor + 1$ different edge colorings. To see this, observe that for a given p, and a given coloring associated with p, c_0 will appear j times at one of the w_i, and $p - j$ times at the other, for some $j \in \{0, \ldots, \lfloor \frac{p}{2} \rfloor\}$.

Two colorings are obviously equivalent if this statement holds for the same value of j, and are different otherwise.

Therefore the number of different colorings is $\sum_{p=1}^{n} (\lfloor \frac{p}{2} \rfloor + 1)$, which is easily seen to equal the value given in the theorem statement. □

The argument in the proof of Theorem 3 counts 3 different colorings of $K_{2,2}$ with 3 colors, but two of these are equivalent, because the vertices of $K_{2,2}$ can be renamed so that the parts in the bipartition switch places.

Theorem 3 is made possible by Corollary 2, which sufficiently specifies the form of a rainbow-C_4-forbidding edge-coloring of $K_{2,n}$ with $n + 1$ colors appearing to make the counting possible. For $3 \leq m \leq n$ we do not have such a useable characterization of the rainbow-C_4-forbidding edge colorings of $K_{m,n}$ with $m + n - 1$ colors appearing. We do, however, have the following, which almost qualifies as a recursive rule for the formation of all such colorings.

Theorem 4. *Suppose that $2 \leq m \leq n$. Every rainbow-cycle-forbidding edge-coloring of $K_{m,n}$ with $m + n - 1$ colors appearing is obtainable by extending a rainbow-cyle-forbidding edge-coloring of $K_{m,n-1}$ with $m + n - 2$ colors appearing.*

Proof. Suppose we have a rainbow-cycle-forbidding edge-coloring of $K_{m,n}$ with $m + n - 1$ colors appearing. By Corollary 1, each vertex of $K_{m,n}$ has at least one color dedicated to it. The n sets of colors dedicated to the vertices in the part of size n are pairwise disjoint. If each one of those sets were to have 2 or more elements, then the total number of colors appearing would be $\geq 2n > m+n-1$. Therefore, at least one of those n vertices, call it v, has exactly one color dedicated to it. Therefore, the coloring restricted to $K_{m,n} - v \simeq K_{m,n-1}$ is rainbow-cycle-forbidding and has $(m+n-1)-1 = m+n-2$ colors appearing. □

References

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