On Subsums of Series with Positive Terms

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A necessary and sufficient condition for subsums of a series \( \sum_{k=1}^{\infty} a_k \), such that \( 0 < a_k \rightarrow 0 \) as \( k \rightarrow \infty \), cover the interval \([0, s]\), \( s = \sum_{k=1}^{\infty} a_k \), is proved.

Introduction

Throughout, \( \mathbb{N} \) will denote the set of non-negative integers. For a series \( \sum_{k=0}^{\infty} a_k \) in which \( a_k \geq 0 \) for all \( k \in \mathbb{N} \), a subsum of the series is a sum of the form \( \sum_{k \in S} a_k \) for some \( S \subseteq \mathbb{N} \).

We use the convention that \( \sum_{k \in \emptyset} a_k = 0 \). If \( 0 < |S| < \infty \), then \( \sum_{k \in S} a_k \) is an ordinary finite subsum. If \( S \) is infinite, then \( \sum_{k \in S} a_k \) is an infinite series in its own right.

We allow \( \infty \) as a series sum. We shall not distinguish between \( \sum_{k \in S} a_k \) as a formal sum and the value of the sum. The set of subsum values of a series \( \sum_{k=0}^{\infty} a_k \) is

\[
\text{Subsum}(a_k) := \left\{ \sum_{k \in S} a_k \mid S \subseteq \mathbb{N} \right\}.
\]

If \( s = \sum_{k=0}^{\infty} a_k \) then \( \text{Subsum}(a_k) \subseteq [0, s] \subseteq [0, \infty]\).

**Theorem 1.** Suppose that \( a_0, a_1, \ldots \) are positive real numbers, \( a_k \rightarrow 0 \) as \( k \rightarrow \infty \), and \( s = \sum_{k=0}^{\infty} a_k \).

1. If \( a_k \leq \sum_{j=k+1}^{\infty} a_j \forall k \in \mathbb{N} \), then

\[
\text{Subsum}(a_k) = \left\{ \sum_{j \in S} a_j \mid S \subseteq \mathbb{N} \right\} = [0, s].
\]

2. If \( a_0 \geq a_1 \geq \ldots \) and \( a_k > \sum_{j=k+1}^{\infty} a_j \) for some \( k \in \mathbb{N} \), then

\[
I = \left( \sum_{j \in \mathbb{N} \setminus \{k\} \cup \{0\}} a_j, \sum_{j \in \mathbb{N} \setminus \{k\}} a_j \right) \text{ is nonempty, and } \text{Subsum}(a_k) \cap I = \emptyset.
\]

Proof. Suppose that \( a_k \leq \sum_{j=k+1}^{\infty} a_j \forall k \in \mathbb{N} \). Clearly 0, \( s \in \text{Subsum}(a_k) \). Suppose that \( x \in (0, s) \). We will define a sequence \( \lambda_0, \lambda_1, \ldots \) such that for each \( k \in \mathbb{N} \), \( \lambda_k \in [0, a_k] \) and \( \sum_{k=0}^{\infty} \lambda_k = x \). From this it will follow that \( x = \sum_{k=0}^{\infty} a_k \) where \( S = \{k \in \mathbb{N} \mid \lambda_k \neq 0\} \).

Having determined \( \lambda_j \) for all \( j < k \), set \( \lambda_k = a_k \) if \( x - \left( \sum_{j<k} \lambda_j + a_j \right) > 0 \) and \( \lambda_k = 0 \), otherwise. It remains to be seen that \( x = \sum_{j=0}^{\infty} \lambda_j \).

Clearly the partial sums \( k \lambda_j \) are non-decreasing and bounded above by \( x \), so we know our series converges to something less than or equal to \( x \).

If \( \lambda_k = 0 \) then \( x - \left( \sum_{j<k} \lambda_j + a_j \right) \leq 0 \), so \( 0 < x - \sum_{j<k} \lambda_j \leq a_k \).

Since \( a_k \rightarrow 0 \), it follows that if \( \lambda_k = 0 \) for infinitely many values of \( k \), then \( \sum_{j=0}^{\infty} \lambda_j = x \).

Otherwise, \( \lambda_k = 0 \) for only finitely many values of \( k \).

Since \( \lambda_k \in [0, a_k] \) for each \( k \in \mathbb{N} \) and \( x < s = \sum_{k=0}^{\infty} a_k \), \( Z = \{k \in \mathbb{N} \mid \lambda_k = 0\} \) is nonempty.

Let \( z \) be the largest element of \( Z \). So \( x - \left( \sum_{j \leq z} \lambda_j + a_i \right) \leq 0 \), and

\[
\sum_{k=0}^{\infty} \lambda_k \leq x \leq \sum_{k=0}^{\infty} \lambda_k + a_i = \sum_{k \leq z} \lambda_k + a_i \leq \sum_{k \leq z} \lambda_k + \sum_{k=z+1}^{\infty} a_k.
\]

Therefore \( \sum_{k=0}^{\infty} \lambda_k \leq x \leq \sum_{k=0}^{\infty} \lambda_k \), so \( x = \sum_{k=0}^{\infty} \lambda_k \), and because \( x \) was chosen arbitrarily from \( (0, s) \), (1) is proven.
Now suppose that $a_0, a_1, \ldots$ is a non-increasing sequence of positive real numbers which converges to 0. Suppose that for some $k \in \mathbb{N}$, $a_k > \sum_{j=k} a_j$. Therefore

$$\sum_{j \in \mathbb{N}[k]} a_j = \sum_{j=k} a_j + \sum_{j<k} a_j < \sum_{j<k} a_j + a_k = \sum_{0 \leq j \leq k} a_j,$$

so $I = \left(\sum_{j \in \mathbb{N}[k]} a_j, \sum_{0 \leq j \leq k} a_j\right)$ is a nonempty open interval contained inside $\left[0, \sum_{j=0}^\infty a_j\right]$, and we will see that

$I \cap \text{Subsum}(a_j) = \emptyset$.

Suppose that $S \subseteq \mathbb{N}$. If some $r \in \{0, \ldots, k\}$ is missing from $S$, then because $a_r \geq a_k$,

$$\sum_{j \in S} a_j \leq \sum_{j \in \mathbb{N}[r]} a_j \leq \sum_{j \in \mathbb{N}[k]} a_j,$$

Otherwise, $\{0, \ldots, k\} \subseteq S$, so $\sum_{j \in S} a_j \leq \sum_{j \in S} a_j$. In either case, $\sum_{j \in S} a_j \not\in I$, so (2) is proven. $\square$

**Corollary 1.** If $a_0 \geq a_1 \geq \cdots > 0$, $a_k \to 0$ as $k \to \infty$, and $s = \sum_{k=0}^{\infty} a_k$, then the finite subsums of the series $\sum_{k=0}^{\infty} a_k$ are dense in $[0, s]$ if and only if $a_k \leq \sum_{j=k+1}^{\infty} a_j$ for all $k \in \mathbb{N}$.

**Corollary 2.** If $a_0, a_1, \ldots$ are positive real numbers tending to 0 such that $|a_i|_{i=0}^{\infty}$ satisfies the hypothesis of (1) of Theorem 1, then so does the non-increasing rearrangement of $|a_i|_{i=0}^{\infty}$.

Let $\zeta$ denote the Riemann zeta function. We have the following:

**Theorem 2.** Suppose that $p \geq 1$. Then

$$\left\{1 + \sum_{k=3}^{\infty} \frac{1}{n^p} \right\}_{S \subseteq \{2, 3, \ldots\}, |S| < \infty}$$

is dense in $[1, \zeta(p)]$ if and only if $p \leq q$, where $q$ (approx. 2.424) is the unique solution of the equation

$$\frac{1}{2^q} = \sum_{k=3}^{\infty} \frac{1}{n^q}.$$

**Proof.** Since $\frac{1}{k} < \sum_{n=k}^{\infty} = \infty$ for all $k = 2, 3, \ldots$, the claim of Theorem 2 for $p = 1$ follows from Corollary 1 applied to the sequence $\frac{1}{2}, \frac{1}{3}, \ldots$. Suppose $p > 1$. For $k \geq 2$,

$$\left(\sum_{n=k}^{\infty} \frac{1}{n^p}\right) / k^p = \sum_{r=0}^{\infty} \left(\frac{k}{k + r + 1}\right)^p = \sum_{r=0}^{\infty} \left(1 - \frac{r + 1}{k + r + 1}\right)^p,$$

which increases as $k$ increases. Therefore, $\frac{1}{k^p} > \sum_{n=k}^{\infty} \frac{1}{n^p}$ for some $k \in \{2, 3, \ldots\}$ if and only if $\frac{1}{2^q} > \sum_{n=3}^{\infty} \frac{1}{n^q}$. Applying Corollary 1 to the sequence $\frac{1}{2}, \frac{1}{3}, \ldots$, we see that

$$\left\{1 + \sum_{n=3}^{\infty} \frac{1}{n^p} \right\}_{S \subseteq \{2, 3, \ldots\}, |S| < \infty}$$

is dense in $[1, \zeta(p)]$ if and only if $\frac{1}{2^q} < \sum_{n=3}^{\infty} \frac{1}{n^q}$. Note that $\sum_{n=3}^{\infty} \frac{1}{n^p} / \frac{1}{2^q} = \sum_{n=3}^{\infty} \left(\frac{1}{n}\right)^p$ decreases as $p$ increases. Since the ratio is large for $p$ close to 1 and goes to 0 (continuously) as $p$ approaches infinity, the equation $\frac{1}{2^q} = \sum_{n=3}^{\infty} \frac{1}{n^q}$ has a unique solution, $q$. Then

$$\frac{1}{2^q} < \sum_{n=3}^{\infty} \frac{1}{n^q}$$

if and only if $p \leq q$, and the theorem is proved. $\square$

Since $\sum_{n=3}^{\infty} \frac{1}{n^q} > \int_3^{\infty} \frac{dx}{x^q} = \frac{1}{2^q} > \frac{1}{2^q}$ and $\sum_{n=3}^{\infty} \frac{1}{n^q} < \int_2^{\infty} \frac{dx}{x^q} = 1/8 = 1/2^3$, $2 < q < 3$. Estimating $q$ by a simple program written in Java, we have $q$ is approximately 2.424.

**Remarks and Open Questions**

The questions that led to the results of this paper arose from Defant (2015), which gives an answer similar to Theorem 2 to a much more difficult question. Here is the question answered:

If we define $\sigma_r$ on the positive integers by $\sigma_r(n) = \sum_{d|r, d>0} d^r$, in which $r \geq 1$, is the range of $\sigma_r$, dense in $[1, \zeta(r)]$? This is a question about a very special set of subsums of $\sum_{k=1}^{\infty} k^{-r}$. The answer (Theorem 2.3 in Defant (2015)) is: the range of $\sigma_r$ is dense in $[1, \zeta(r)]$ for $1 \leq r \leq \kappa$, where $\kappa$ is the unique solution in (1, 2) of the equation $\frac{\zeta}{\zeta - 1} = \zeta(\kappa)$ (Defant estimates $\kappa \approx 1.888$).

We end with two questions: suppose that $a_0 \geq a_1 \geq \cdots > 0$, $a_k \to 0$ as $k \to \infty$, and $s = \sum_{k=0}^{\infty} a_k$.

1. Is $\text{Subsum}(a_k)$ necessarily closed? By Theorem 1, the answer is obviously yes if $a_k \leq \sum_{j=k+1} a_j$ for every $k \in \mathbb{N}$. It follows that it is true if this holds for all but finitely many $k$.

2. Can there be any maximal open intervals in $[0, s] \setminus \text{Subsum}(a_k)$ other than the intervals $\left(\sum_{j \in \mathbb{N}[k]} a_j, \sum_{j \leq k} a_j\right)$ for $k \in \mathbb{N}$ such that $a_k > \sum_{j=k+1}^{\infty} a_j$?

**References**